

# Improved Concentration Bounds for Count-Sketch

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## Abstract

We present a refined analysis of the classic Count-Sketch streaming heavy hitters algorithm [CCF02]. Count-Sketch uses  $O(k \log n)$  linear measurements of a vector  $x \in \mathbb{R}^n$  to give an estimate  $\hat{x}$  of  $x$ . The standard analysis shows that this estimate  $\hat{x}$  satisfies  $\|\hat{x} - x\|_\infty^2 < \|x_{\overline{[k]}}\|_2^2/k$ , where  $x_{\overline{[k]}}$  is the vector containing all but the largest  $k$  coordinates of  $x$ . Our main result is that most of the coordinates of  $\hat{x}$  have substantially less error than this upper bound; namely, for any  $c < O(\log n)$ , we show that each coordinate  $i$  satisfies

$$(\hat{x}_i - x_i)^2 < \frac{c}{\log n} \cdot \frac{\|x_{\overline{[k]}}\|_2^2}{k}$$

with probability  $1 - 2^{-c}$ , as long as the hash functions are fully independent. This subsumes the previous bound. Our improved point estimates also give better results for  $\ell_2$  recovery of exactly  $k$ -sparse estimates  $x^*$  when  $x$  is drawn from a distribution with suitable decay, such as a power law.

Our proof shows that any random variable with positive real Fourier transform and finite variance concentrates around 0 at least as well as a Gaussian. This result, which may be of independent interest, gives good concentration even when the noise does not converge to a Gaussian.

## 1 Introduction

The *heavy hitters* problem and the closely related *sparse recovery* problem are two of the most fundamental problems in the field of sketching and streaming algorithms [GI10, CH10, Mut05]. The goal is to efficiently identify and estimate the  $k$  largest coordinates of an  $n$ -dimensional vector using a linear sketch  $Ax$  of  $x$ , where  $A \in \mathbb{R}^{m \times n}$  has  $m = O(k \log^c n)$  rows. The strongest commonly used formal guarantee for the quality of such an estimate is the  $\ell_\infty/\ell_2$  guarantee: this is a bound for the estimate  $\hat{x}$  recovered from  $Ax$  which is of the form

$$\|\hat{x} - x\|_\infty^2 \leq \|x_{\overline{[k]}}\|_2^2/k, \tag{1}$$

where  $x_{\overline{[k]}}$  denotes the vector obtained from  $x$  by replacing its largest  $k$  coordinates with 0.

The classic approach for this problem is the Count-Sketch algorithm of Charikar et al. [CCF02], which uses  $m = O(k \log n)$  measurements and satisfies (1) with  $1 - 1/n^{\Theta(1)}$  probability. It is simple, practical, and gives the best known theoretical performance in many settings. This paper shows that the quality of the approximation  $\hat{x}$  given by Count-Sketch is much better than the standard bound (1) suggests. While (1) gives a bound on the *worst-case* error of  $\hat{x}$ , we prove that *most* coordinates of  $\hat{x}$  have a  $1/\sqrt{\log n}$  factor less error than this worst case.

The Count-Sketch of a vector  $x$  using  $R$  rows of  $C$  columns is defined as follows. For  $u \in [R]$ , we choose hash functions  $h_u : [n] \rightarrow [C]$  and  $s_u : [n] \rightarrow \{\pm 1\}$ . The sketch is

$$y_{u,v} = \sum_{i:h_u(i)=v} s_u(i)x_i,$$

which consists of  $RC$  linear measurements. The estimate  $\hat{x}$  is given by

$$\hat{x}_i = \text{median}_u s_u(i)y_{u,h_u(i)}.$$

Setting  $C = O(k)$  and  $R = O(\log n)$ , [CCF02] proves that (1) holds with  $1 - 1/n^{\Theta(1)}$  probability. As per [CM06, GI10], the  $\ell_\infty/\ell_2$  guarantee can be converted into an  $\ell_2/\ell_2$  guarantee: if  $x^*$  contains the largest  $2k$  coordinates of  $\hat{x}$ , then

$$\|x^* - x\|_2^2 \leq O(\|x_{[k]}\|_2^2).$$

Our main result is the following strengthening of the analysis in [CCF02] for the concentration of the point estimates  $\hat{x}_i$  resulting from Count-Sketch, assuming the hash functions are fully random:

**Main Result (Theorem 4.1).** *Consider the estimate  $\hat{x}$  of  $x$  from Count-Sketch using  $R$  rows and  $k \geq 2$  columns, with fully random hash functions. For any  $t \leq R$  and each index  $i$ ,*

$$\Pr \left[ (\hat{x}_i - x_i)^2 > \frac{t}{R} \cdot \frac{\|x_{[k]}\|_2^2}{k} \right] < 2e^{-\Omega(t)}.$$

The standard analysis [CCF02] gives the special case of  $t = R$ . One gets (1) by setting  $t = R = \Theta(\log n)$  and applying a union bound. Our result says that most coordinates of  $\hat{x}$  have much better estimates than this worst-case bound. It gives no direct improvement to (1), but this is expected: an improved  $\ell_\infty/\ell_2$  bound would allow for improved  $\epsilon$  dependence in  $\ell_2/\ell_2$  recovery, which is impossible [PW11].

Although an improvement in  $\ell_2$  reconstruction is impossible for general vectors  $x$ , for some common distributions on  $x$  it is possible. For example, if  $x$  follows the power law  $x_i = i^{-\alpha}$  for some constant  $\alpha > 0.5$ , Theorem 4.1 implies that we can reconstruct a  $k$ -sparse  $x^*$  from  $\hat{x}$  satisfying

$$\|x^* - x\|_2^2 < (1 + \frac{1}{\log n}) \|x_{[k]}\|_2^2$$

with constant probability (see Theorem 4.2). This beats the  $1 + \epsilon$  approximation of traditional analysis. Previous work [Pri11] combined the Count-Sketch with another sketch to get a  $(1 + \frac{1}{\sqrt{\log n}})$  factor approximation for this problem.

Our analysis requires that the hash functions be fully random. This is unfortunate because fully random hash functions would take up more space than the sketch itself, but there are some reasons why this constraint is not too problematic. One reason is that Nisan's pseudorandom number generator [Nis92] lets us store the hash functions with only a  $\log n$  factor increase in space. Then if we wish to run Count-Sketch on multiple different vectors, we can reuse the hash functions. A second reason is that one expects bounded independence to suffice as long as the vector  $x$  itself has sufficient entropy. A result of this form is known [MV08] when  $\text{supp}(x)$  is drawn at random from a much larger domain. For example, if  $\text{supp}(x)$  contains  $n^{1/3}$  random coordinates out of  $n$ , then [MV08] implies near-uniformity with 4-wise independence.

**Our Techniques** Our basic strategy is to translate the problem of bounding Count-Sketch error into a problem of proving a strong concentration result for a certain class of random variables. This, in turn, we solve by analyzing the Fourier transform of such variables.

In more detail, the argument proceeds as follows. The error  $\hat{x}_i - x_i$  is, by definition, the median over rows of error terms coming from the different coordinates which hash to the same column as  $i$ . For each row, we separate the error term into contributions from (i) the largest  $k$  coordinates  $j \in [k]$  and (ii) the remaining coordinates  $j \in \overline{[k]}$ . The error of type (i) is zero with constant probability, and we bound the error of type (ii) with our concentration result. We then get a bound on  $\hat{x}_i - x_i$  by using Chernoff bounds to conclude that if each of  $R$  symmetric random variables has a  $\sqrt{c/R}$  chance of being small, then the median has a  $1 - 2e^{-c/2}$  chance of being small.

The concentration result we prove is a bound of the form  $\Pr[|X| < \epsilon] > \Omega(\epsilon)$ , where  $X$  has variance 1 and is a sum of independent random variables, each of which is symmetric and zero with probability at least  $1/2$ . Such a bound certainly holds in the limit as  $X$  converges to a Gaussian, but we need it to be true before  $X$  converges. To see why this is subtle, consider the sum of  $n$  independent  $\pm 1/\sqrt{n}$  variables. The Berry-Esséen theorem gives our bound for  $\epsilon > 1/\sqrt{n}$ , but the bound is actually false for  $\epsilon < 1/\sqrt{n}$  when  $n$  is odd. When  $n$  is even, we can pair up the variables to get  $n/2$  independent  $\{0, \pm 2/\sqrt{n}\}$  variables. These variables are zero with  $1/2$  probability, so our bound applies for arbitrarily small  $\epsilon$ . What distinguishes even  $n$  from odd  $n$ ?

The key for our argument is that, for a symmetric random variable  $X$  with at least  $1/2$  probability of being 0, the Fourier transform of  $X$  is *nonnegative*. The Fourier transform of the triangle filter  $\max\{1 - |x|/\epsilon, 0\}$  is also nonnegative. We use the convolution theorem to translate the expectation of the triangle filter into an integral in Fourier space, and then use positivity to note that we can bound that integral over all Fourier space by the integral over small frequencies. This we control directly by using the quadratic Taylor series approximation to  $\cos x$ . Because a lower bound on the expectation of the triangle filter also gives a lower bound on  $\Pr[|X| < \epsilon]$ , this proves what we want.

**Related Work** Count-Min and Count-Median [CM04] are similar algorithms to Count-Sketch that get the weaker  $\ell_\infty/\ell_1$  guarantee. Their sketch matrix is the same as Count-Sketch's but with  $s_u(i) = 1$  always. This means that the random variables involved in the error are not symmetric, so the estimates do not have the additional concentration that we show in this paper.

The application of heavy hitters algorithms to power law distributions was studied in [CM05].

## 2 Preliminaries

**Notation** We use  $f \gtrsim g$  to denote  $f = \Omega(g)$  and  $f \lesssim g$  to denote  $f = O(g)$ .

In the statement of Theorem 4.1,  $x_{\overline{[k]}}$  denotes the vector consisting of all but the largest  $k$  coordinates of  $x$ . More generally, we think of the coordinates of  $x$  as being sorted,  $|x_1| \geq |x_2| \geq \dots \geq |x_n|$ . This is purely a notational convenience, possible because Count-Sketch is invariant under permutation of coordinates.

Given a real-valued random variable  $X$ , its *Fourier transform* is the function

$$\mathcal{F}(t) = \mathbb{E}[e^{2\pi X t \sqrt{-1}}].$$

In general  $\mathcal{F}(t)$  is complex-valued. However, our random variables are all symmetric; in this case  $\mathcal{F}(t)$  is real-valued and equals  $\mathbb{E}[\cos(2\pi X t)]$ .

### 3 Concentration Lemmas

The following is the key lemma for our proof.

**Lemma 3.1.** *Let  $X$  be a symmetric, real-valued random variable with variance 1, and suppose that its Fourier transform  $\mathcal{F}(t)$  is nonnegative. Then, for  $\epsilon < 1$ ,  $\Pr[|X| < \epsilon] \gtrsim \epsilon$ .*

*Proof.* Because  $\cos x \geq 1 - \frac{1}{2}x^2$  holds for all  $x \in \mathbb{R}$ , we have

$$\mathcal{F}(t) \geq \mathbb{E}[1 - \frac{1}{2}(2\pi Xt)^2] = 1 - 2\pi^2 t^2 \quad \forall t \in \mathbb{R}.$$

In particular,  $\mathcal{F}(t) \geq \frac{1}{2}$  for  $t \in [-\frac{1}{2\pi}, \frac{1}{2\pi}]$ . Let  $T_\epsilon(x)$  be the triangle filter

$$T_\epsilon(x) = \begin{cases} 1 - \frac{1}{\epsilon}|x| & \text{if } |x| < \epsilon \\ 0 & \text{otherwise} \end{cases}$$

and recall the Fourier transform relation

$$T_\epsilon(x) = \int_{-\infty}^{\infty} \frac{\sin^2(\pi t \epsilon)}{\pi^2 t^2 \epsilon} e^{2\pi i x t \sqrt{-1}} dt.$$

Using this relation and switching the order of integration,  $\mathbb{E}[T_\epsilon(X)] = \int_{-\infty}^{\infty} \frac{\sin^2(\pi t \epsilon)}{\pi^2 t^2 \epsilon} \mathcal{F}(t) dt$ . The integrand is nonnegative, so we get a lower bound on  $\mathbb{E}[T_\epsilon(X)]$  by integrating only over the interval  $[-\frac{1}{2\pi}, \frac{1}{2\pi}]$ . On this interval we have  $\mathcal{F}(t) \geq \frac{1}{2}$  and, because  $\epsilon < 2\pi$ ,  $\frac{\sin^2(\pi t \epsilon)}{\pi^2 t^2 \epsilon}$  is bounded below by its value at  $t = 1/(2\pi)$ . Putting this together, we find that

$$\mathbb{E}[T_\epsilon(X)] \geq \int_{-1/(2\pi)}^{1/(2\pi)} \frac{\sin^2(\pi t \epsilon)}{\pi^2 t^2 \epsilon} \mathcal{F}(t) dt \gtrsim \frac{\sin^2(\epsilon/2)}{\epsilon}.$$

For  $\epsilon < 1$  we have  $\frac{\sin^2(\epsilon/2)}{\epsilon} \gtrsim \epsilon$ . Now noting that  $\Pr[|X| < \epsilon] \geq \mathbb{E}[T_\epsilon(X)]$  completes the proof.  $\square$

**Corollary 3.2.** *Let  $\{X_i : i \in [n]\}$  be independent symmetric random variables such that  $\Pr[X_i = 0] \geq 1/2$  for each  $i$ . Set  $X = \sum_{i=1}^n X_i$  and  $\sigma^2 = \mathbb{E}[X^2]$ . For  $\epsilon < 1$ ,  $\Pr[|X| < \epsilon\sigma] \gtrsim \epsilon$ .*

*Proof.* For each  $i \in [n]$ , let  $p_i = \Pr[X_i = 0]$ . The Fourier transform of  $X_i$  is  $\mathcal{F}_i(t) = p_i + (1 - p_i) \mathbb{E}[\cos(2\pi X_i t) \mid X_i \neq 0] \geq p_i + (1 - p_i)(-1)$ . Because  $p_i \geq 1/2$ , this is nonnegative. Now  $X/\sigma$  is a symmetric random variable with nonnegative Fourier transform  $\prod_{i=1}^n \mathcal{F}_i(t/\sigma)$  and with variance  $\mathbb{E}[(X/\sigma)^2] = 1$ ; applying Lemma 3.1 to it gives the desired bound.  $\square$

Note that Lemma 3.1 is not true without the positivity assumption, and in particular Corollary 3.2 is not true when  $\Pr[X_i = 0]$  is small. Indeed, it seems intuitive that we get strong concentration around 0 as a consequence of the large probability of each individual variable being 0. We also remark that there are analogs of Lemma 3.1 and Corollary 3.2 using only first moment bounds. The proof is nearly identical, so we omit it.

We also need the following lemma for concentration of medians.

**Lemma 3.3.** *Suppose  $X_1, \dots, X_t$  are independent symmetric random variables such that, for some  $r, c > 0$ , we have  $\Pr[|X_i| < r] > \sqrt{c/t}$  for all  $i \in [t]$ . Then*

$$\Pr \left[ \left| \text{median}_{i \in [t]} X_i \right| \geq r \right] < 2e^{-c/2}.$$

*Proof.* Let  $E_i$  denote the indicator for the event that  $X_i \geq r$ . Then  $\Pr[E_i = 1] < (1 - \sqrt{c/t})/2$ , so  $\mathbb{E}[\sum_{i=1}^t E_i] < t/2 - \sqrt{ct}/2$ . The  $E_i$  are independent, so by a Chernoff bound we have that

$$\Pr \left[ \sum_{i=1}^t E_i \geq \frac{t}{2} \right] < e^{-2(\sqrt{ct}/2)^2/t} = e^{-c/2}.$$

The same bound applies to the event that at least  $t/2$  of the  $X_i$  are less than  $-r$ , and if neither event occurs then the median is in the interval  $(-r, r)$ .  $\square$

## 4 Count-Sketch

**Theorem 4.1.** *Consider the estimate  $\hat{x}$  of  $x$  from Count-Sketch using  $R$  rows and  $k \geq 2$  columns, with fully random hash functions. For any  $t \leq R$  and each index  $i$ ,*

$$\Pr \left[ (\hat{x}_i - x_i)^2 > \frac{t}{R} \cdot \frac{\|x_{[k]}\|_2^2}{k} \right] < 2e^{-\Omega(t)}.$$

*Proof.* Fix  $i \in [n]$ . For each row  $u$  and coordinate  $j \in [n]$ , define

$$X_{u,j} = \begin{cases} s_u(j)x_j & \text{if } h_u(j) = h_u(i) \\ 0 & \text{otherwise.} \end{cases}$$

For each row  $u$ , define

$$T_u = \sum_{j \in [k] \setminus \{i\}} X_{u,j} \quad \text{and} \quad H_u = \sum_{j \in [k] \setminus \{i\}} X_{u,j}.$$

Then, by definition,

$$\hat{x}_i - x_i = \text{median}_u H_u + T_u.$$

Each random variable  $X_{u,j}$  is symmetric, equals 0 with probability  $1 - 1/k \geq 1/2$ , and otherwise equals  $\pm x_j$ . Moreover, for each row  $u$ , the random variables  $\{X_{u,j}\}$  are independent. Thus Corollary 3.2 shows that

$$\Pr \left[ |T_u| < \epsilon \cdot \frac{\|x_{[k]}\|_2}{\sqrt{k}} \right] \gtrsim \epsilon.$$

Furthermore,  $H_u = 0$  with probability at least  $(1 - 1/k)^k \geq 1/4$ , i.e., with constant probability. Since  $H_u$  is independent of  $T_u$ , this means that

$$\Pr \left[ |H_u + T_u| < \epsilon \cdot \frac{\|x_{[k]}\|_2}{\sqrt{k}} \right] \gtrsim \epsilon.$$

Therefore Lemma 3.3 implies

$$\Pr \left[ |\hat{x}_i - x_i| > \epsilon \cdot \frac{\|x_{[k]}\|_2}{\sqrt{k}} \right] < 2e^{-\Omega(R\epsilon^2)}.$$

Setting  $\epsilon = \sqrt{t/R}$  yields the desired result.  $\square$

As an application to  $\ell_2$  reconstruction, we give the following theorem. Note that the condition  $|x_k| - |x_{2k}| \gtrsim \|x_{[k]}\|_2/\sqrt{k}$  is satisfied by any power law distribution  $x_i = i^{-\alpha}$  with  $\alpha > 0.5$ .

**Theorem 4.2.** Suppose  $|x_k| - |x_{2k}| \gtrsim \|x_{\overline{[k]}}\|_2/\sqrt{k}$ . Let  $\hat{x}$  be the result of Count-Sketch using  $R = O(\log n)$  rows and  $O(k)$  columns, with fully random hash functions. Let  $x^*$  be the vector containing the largest  $k$  entries of  $\hat{x}$ . Then

$$\|x - x^*\|_2^2 < (1 + \frac{1}{R})\|x_{\overline{[k]}}\|_2^2.$$

with  $3/4$  probability.

*Proof.* Let the number of columns be  $ck$  for some constant  $c$ . Let  $S \subset [n]$  contain the largest  $k$  coordinates of  $\hat{x}$ . By the standard Count-Sketch bound we have with  $n^{-\Theta(1)}$  probability that  $\|\hat{x} - x\|_\infty^2 < \|x_{\overline{[ck]}}\|_2^2/(ck) =: \mu^2$ . Then for sufficiently large  $c$ ,  $|\hat{x}_i| > |\hat{x}_j|$  for all  $i \in [k]$  and  $j \in \overline{[2k]}$ , so  $S \subseteq [2k]$ .

Since  $S \subseteq [2k]$ , we have

$$\begin{aligned} \|\hat{x}_S - x\|_2^2 &= \|\hat{x}_S - x_S\|_2^2 + \|x_{\overline{S}}\|_2^2 \\ &\leq \|\hat{x}_{[2k]} - x_{[2k]}\|_2^2 + \|x_{\overline{[k]}}\|_2^2. \end{aligned}$$

For each  $i \in [2k]$ , define  $V_i = \min((\hat{x}_i - x_i)^2, \mu^2)$ . We have by Theorem 4.1 and  $V_i \leq \mu^2$  that

$$\Pr[V_i > \frac{t}{R}\mu^2] < 2e^{-\Omega(t)}$$

for all  $t$ . It follows that  $\mathbb{E}[V_i] \lesssim \mu^2/R$ , so  $\mathbb{E}[\sum_{i \in [2k]} V_i] \lesssim 2k\mu^2/R$ . By Markov's inequality, with constant  $(7/8, \text{ say})$  probability,

$$\sum_{i \in [2k]} V_i \lesssim k\mu^2/R = \frac{1}{cR}\|x_{\overline{[ck]}}\|_2^2 \leq \frac{1}{cR}\|x_{\overline{[k]}}\|_2^2.$$

Now, conditioned on  $\|\hat{x} - x\|_\infty^2 < \mu^2$ ,  $V_i = (\hat{x}_i - x_i)^2$ . Thus with at least  $7/8 - n^{-\Theta(1)} > 3/4$  probability, and for sufficiently large  $c$ , we have  $\|\hat{x}_{[2k]} - x_{[2k]}\|_2^2 \leq \frac{1}{R}\|x_{\overline{[k]}}\|_2^2$ , whence

$$\|\hat{x}_S - x\|_2^2 \leq (1 + \frac{1}{R})\|x_{\overline{[k]}}\|_2^2. \quad \square$$

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